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## **SU(2) $\times$ SU(2) $\times$ U(1) basis for symmetric SO(6) representations : matrix elements of the generators (\*)**

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**Résumé.** — Nous calculons de façon explicite et analytique, les éléments de matrice des générateurs du groupe SO(6) pour la représentation irréductible symétrique ; nous employons la chaîne de décomposition  $SO(6) \supset SU(2) \times SU(2) \times U(1)$  (qui est différente du schéma très connu des supermultiplets de Wigner). Nous indiquons la relation avec la méthode de Gel'fand et Tsetlin qui utilise la chaîne  $SO(6) \supset SO(5) \supset \dots \supset SO(2)$ . Nous présentons aussi un exemple d'application physique.

**Abstract.** — Matrix elements of the group generators for the symmetric irreducible representations of SO(6) are explicitly calculated in a closed form employing the decomposition chain  $SO(6) \supset SU(2) \times SU(2) \times U(1)$  (which is different from the well known Wigner supermultiplet scheme). The relation to the Gel'fand Tsetlin method using  $SO(6) \supset SO(5) \supset \dots \supset SO(2)$  is indicated. An example of a physical application is given.

### **1. Introduction.**

The algebra of the rotation group in 6 dimensions SO(6), equivalent to that of SU(4), plays an important role in nuclear and particle physics. The decomposition chain  $SO(6) \supset SU(2) \times SU(2)$  has been widely known as the Wigner supermultiplet scheme (see e.g. Wigner [1] or Hecht and Pang [2]). Another well-known decomposition of the irreducible representations of SO( $n$ ) (and in particular of SO(6)) has been provided by the work of Gel'fand and Tsetlin [3]).

However, in some cases of physical interest (see e.g. Krumlinde and Szymanski [4] or Piepenbring *et al.* [5]) a decomposition different from the above mentioned Wigner supermultiplet scheme is needed. It is based on the embedding  $SO(6) \supset SO(4) \times U(1) \sim SU(2) \times SU(2) \times U(1)$ . It is a particular case of decompositions of Lie groups into products involving the U(1) subgroup as discussed in references [6, 7]. Relevant decompositions have also been reviewed in references [8-10].

In the present paper we intend to give a complete and elementary derivation of the explicit formulae for the matrix elements of the SO(6) group

generators in the symmetric irreducible representation. In section 2 we establish the tensorial character of the 15 group generators with respect to  $SU(2) \times SU(2) \times U(1)$ . Sections 3 to 5 are devoted to a discussion of the symmetric irreducible representations of SO(6). In section 6 we explicitly calculate in a closed form the matrix elements of the group generators for the symmetric representations of SO(6) employing the decomposition chain  $SO(6) \supset SU(2) \times SU(2) \times U(1)$ . In section 7 we indicate the relations of the obtained basis with that of Gel'fand Tsetlin. Finally, section 8 brings an example of a physical application and section 9 the conclusions.

### **2. Group generators and their tensorial classification.**

The 15 generators of the SO(6) Lie algebra can be defined as the operators of generalised momenta  $J_{ab}$  with  $a, b = 1, 2, \dots, 6$  (cf. for example [2, 4]). In the case of the Wigner supermultiplet decomposition the two SU(2) subgroups are simply formed by  $J_{ab}$  with  $a, b = 1, 2, 3$  and  $J_{ab}$  with  $a, b = 4, 5, 6$ , respectively. The Cartan subalgebra may be chosen as defined by the three operators  $J_{12}$ ,  $J_{34}$  and  $J_{56}$ . This scheme, however, has the disadvantage that none of these three operators commute simultaneously with all the generators forming the two subgroups SU(2). For our purpose it is, therefore,

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more convenient to choose first the four-dimensional subspace (say, corresponding to  $J_{ab}$  with  $a, b = 1, 2, 3, 4$ ). It is well-known that the resulting  $SO(4)$  subgroup is equivalent to the product  $SU(2) \times SU(2)$  of the two commuting ordinary angular momenta. We shall call them quasispins as to maintain the analogy with the typical application of section 8 (see also [5]).

Now, the operator  $J_{56}$  commutes with all the six generators of the  $SO(4)$  subgroup and can thus define a  $U(1)$  subgroup completing our chain  $SO(6) \supset SU(2) \times SU(2) \times U(1)$ .

In general (see Racah [11]) the states forming an irreducible representation basis of the group  $SO(6)$  can be completely characterized by specifying  $\frac{1}{2}(p-r) = 6$  quantum numbers in addition to the  $r$  quantum numbers specifying the representation itself (by its highest weight). Here  $p = 15$  denotes the number of generators of  $SO(6)$  while  $r = 3$  is the rank of the group (see e.g. [2, 11]). However, we shall show that, in the case of a symmetric representation, four quantum numbers will be sufficient to determine the representation basis completely.

We shall now define explicitly all the 15 generators of  $SO(6)$  in terms of the generalised angular momenta  $J_{ab}$  ( $a, b = 1, 2 \dots 6$ ). The two quasispins, say,  $\mathbf{K}$  and  $\mathbf{L}$  can be introduced as

$$K_{\pm} = -\frac{1}{2} \{ (J_{13} + J_{24}) \pm i (J_{23} - J_{14}) \} \quad (1)$$

$$K_0 = \frac{1}{2} (J_{12} - J_{34})$$

and

$$L_{\pm} = -\frac{1}{2} \{ (J_{13} - J_{24}) \pm i (J_{23} + J_{14}) \} \quad (2)$$

$$L_0 = \frac{1}{2} (J_{12} + J_{34}).$$

It is easy to see that the two quasispins commute, i.e.

$$[\mathbf{K}, \mathbf{L}] = 0. \quad (3)$$

It is also obvious that the operator  $J_{56}$  commutes with the 6 components of the two quasispins  $\mathbf{K}$  and  $\mathbf{L}$ . The three mutually commuting operators  $J_{12} = K_0 + L_0$ ,  $J_{34} = -(K_0 - L_0)$  and  $J_{56}$  are chosen to form the Cartan subalgebra. The remaining 8 generators of  $SO(6)$  can be arranged as tensors of the order  $\frac{1}{2}$  with respect to both the quasispins  $\mathbf{K}$  and  $\mathbf{L}$ . Simultaneously, these tensors carry  $+1$  or  $-1$  of the quantity  $d$  which is defined as an eigenvalue of  $(-J_{56})$ . In other words

$$[K_{\pm}, T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}] = \sqrt{\left(\frac{1}{2} \mp \kappa\right) \left(\frac{1}{2} \pm \kappa + 1\right)} T_{\kappa \pm 1 \ \lambda}^{1/2 \ 1/2 \ \delta} \quad (4a)$$

$$[K_0, T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}] = \kappa T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta} \quad (4b)$$

$$[L_{\pm}, T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}] = \sqrt{\left(\frac{1}{2} \mp \lambda\right) \left(\frac{1}{2} \pm \lambda + 1\right)} T_{\kappa \ \lambda \pm 1}^{1/2 \ 1/2 \ \delta} \quad (4c)$$

$$[L_0, T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}] = \lambda T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta} \quad (4d)$$

and

$$[J_{56}, T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}] = -\delta T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}. \quad (4e)$$

The choice of the quantum number  $d$  as an eigenvalue of  $(-J_{56})$  (instead of  $J_{56}$ ) has been made as to remain consistent with the results of [5] where  $d$  has a simple physical meaning <sup>(1)</sup>.

Closed-form expressions of the eight components  $T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}$  in terms of the generalised angular momenta  $J_{ab}$  can be given explicitly. In order to simplify the notation  $T_{\kappa \lambda}^{k \ l \ \delta}$  we shall omit the superscripts  $k$  and  $l$  that are always equal to  $1/2$  and replace quantum numbers  $\kappa$  ( $= \pm 1/2$ ),  $\lambda$  ( $= \pm 1/2$ ) and  $\delta$  ( $= \pm 1$ ) by their signs. Thus for example  $T_{-1/2 \ 1/2}^{1/2 \ 1/2 \ -1}$  will be replaced by  $T_{-+}$ , etc.

$$T_{++} = \frac{1}{2} \{ (J_{25} - J_{16}) - i (J_{26} + J_{15}) \} \quad (5a)$$

$$T_{+-} = -\frac{1}{2} \{ (J_{35} - J_{46}) - i (J_{36} + J_{45}) \} \quad (5b)$$

$$T_{-+} = -\frac{1}{2} \{ (J_{46} + J_{35}) + i (J_{45} - J_{36}) \} \quad (5c)$$

$$T_{--} = -\frac{1}{2} \{ (J_{25} + J_{16}) - i (J_{26} - J_{15}) \} \quad (5d)$$

and

$$T_{+-} = \frac{1}{2} \{ (J_{25} + J_{16}) + i (J_{26} - J_{15}) \} \quad (5e)$$

$$T_{-+} = -\frac{1}{2} \{ (J_{46} + J_{35}) - i (J_{45} - J_{36}) \} \quad (5f)$$

$$T_{--} = -\frac{1}{2} \{ (J_{35} - J_{46}) + i (J_{36} + J_{45}) \} \quad (5g)$$

$$T_{--} = -\frac{1}{2} \{ (J_{25} - J_{16}) + i (J_{26} + J_{15}) \}. \quad (5h)$$

Tensors  $T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta}$  obey the following hermiticity condition

$$(T_{\kappa \lambda}^{1/2 \ 1/2 \ \delta})^{\dagger} = (-1)^{\kappa + \lambda} T_{-\kappa \ -\lambda}^{1/2 \ 1/2 \ -\delta}. \quad (6)$$

The eight generators  $T_{\kappa \lambda}^{k \ l \ \delta}$  with  $k = l = \frac{1}{2}$  and  $\delta = \pm 1$  exhibit the difference of the present classification as compared with the Wigner super-multiplet scheme [2] where the nine analogous generators  $T_{\kappa \lambda}^{k \ l}$  are characterized by  $k = l = 1$ .

<sup>(1)</sup> Throughout this paper, we use the symbol  $d$  to denote the eigenvalue of  $-J_{56}$ , which can take any integer value satisfying relation (25). The symbol  $\delta$ , which is an index of the tensors we introduce, takes only the two values  $\pm 1$ , and allows a change of  $d$  to  $d + \delta$ .

### 3. Symmetric representations.

The irreducible representations are characterised by their highest weights (see e.g. [11, 12]). The representations may be labelled by the vector  $|J'_{12}, J'_{34}, J'_{56}\rangle$  of the three eigenvalues of the generators  $J_{12}$ ,  $J_{34}$  and  $J_{56}$  forming the Cartan subalgebra. Alternatively, one can employ partitions  $[\lambda] = [\lambda_1 \lambda_2 \lambda_3]$  of an integer number  $l$  ( $l = \lambda_1 + \lambda_2 + \lambda_3$ ) specifying the corresponding Young diagram (see e.g. [12, 13]). The symmetric irreducible representations correspond to the Young diagrams  $[l]$  with one row only, i.e.  $\lambda_1 = \Omega = l$ ,  $\lambda_2 = \lambda_3 = 0$  (with the highest weight corresponding to the vector with components  $\Omega, 0, 0$  in the Cartan space). The positive integer  $\Omega$  may have a certain physical meaning as it will become clear below.

To specify the representation basis, it seems natural to use the two commuting quasispins  $\mathbf{K}$  and  $\mathbf{L}$  together with the eigenvalue  $d$  of  $(-J_{56})$ . Thus we introduce vectors

$$|K, L; M_K, M_L; d\rangle \quad (7)$$

with  $M_K$  denoting the eigenvalue of  $K_0$  defined by equation (1) and  $K$ , the total quasispin  $K$  quantum number (i.e.  $K(K+1)$  is the eigenvalue of  $\mathbf{K}^2$ ). The meanings of  $M_L$  and  $L$  are analogous to  $M_K$  and  $K$  except that they refer to the quasispin  $\mathbf{L}$  (Eq. (2)).

In general, one more quantum number is needed in order to specify the basis vectors of the irreducible representations of SO(6) as already mentioned in the preceding section. However, in the case of the symmetric irreducible representation even the basis (7) is too general since the symmetry of the representations implies that

$$K = L. \quad (8)$$

In order to demonstrate the existence of this constraint we can employ the branching rule

$$[\lambda] \downarrow \sum_{s,t} [\lambda/(s.t)] \times \{s-t\} \quad (9)$$

corresponding to the embedding

$$\text{SO}(2k) \supset \text{SO}(2k-2) \times \text{U}(1) \quad (10)$$

given by King [6]. Here  $(s.t)$  corresponds to the outer product of two Schur functions related to the two one-row Young diagrams, while  $[\lambda/(s.t)]$  denotes the outer division  $[\lambda]/[s.t]$  of two Schur functions (e.g. [12, 14, 15]). Finally  $\{s-t\}$  denotes the representations of U(1) specified by the integer  $s-t$ . For the symmetric irreducible representation  $[l]$  of SO(6) the branching rule (9) is simply reduced to

$$[l] \downarrow \sum_{s,t} [l-s-t] \times \{s-t\} \quad (11)$$

implying that the irreducible representations

$[l-s-t]$  of the subgroup  $\text{SO}(2k-2)$  are also symmetric. In the case of  $k=3$ , one gets SO(4) which is isomorphic to  $\text{SU}(2) \times \text{SU}(2)$ . The symmetry of the SO(4) representation requires the vanishing of its second label, i.e.  $K-L=0$  and hence equation (8).

It follows from equation (8) that we may characterise the basis states of the symmetric representations (7) by only four independent quantum numbers

$$|K, K; M_K, M_L; d\rangle = |K, L; M_K, M_L; d\rangle_{K=L} \quad (12)$$

in addition to the highest-weight labels  $(\Omega, 0, 0)$  characterising the representation itself. In the case of a general irreducible representation of SO(6) we have of course  $L \neq K$  and there remains one more quantum number to specify the representation basis (cf. an analogous problem in the Wigner supermultiplet scheme is discussed by Hecht and Pang [2]).

### 4. Step down operators.

We shall now demonstrate an explicit construction of all the states forming a representation basis. Acting with operators  $K_+$  and  $L_+$  and starting with an arbitrary state  $|K, K; M_K, M_L; d\rangle$  we may always reach the  $M_K = M_L = K$  state which we call hereafter the maximum aligned (MA) state for a given  $K$ :

$$|K, K; M_K = K, M_L = K; d\rangle = |\{K\}; d\rangle \quad (13)$$

where the right hand side is a convenient short-hand notation for the MA states. Thus in order to obtain any state belonging to a symmetric representation basis it is sufficient to construct all possible MA state-vectors. One can show that the two operators

$$\begin{aligned} \sigma_\delta = & -T_{--}^\delta (2K_0 + 1)(2L_0 + 1) + \\ & + K_- T_{+-}^\delta (2L_0 + 1) \\ & + L_- T_{-+}^\delta (2K_0 + 1) - K_- L_- T_{++}^\delta \end{aligned} \quad (14)$$

with  $\delta = +1$  or  $\delta = -1$  are the step-down operators that by acting on the MA states lower the four quantum numbers  $K, L = K, M_K = K$  and  $M_L = K$  by one half. Thus we have

$$\sigma_{\pm 1} |\{K\}; d\rangle = c \left| \left\{ K - \frac{1}{2} \right\}; d \pm 1 \right\rangle \quad (15)$$

where the constant  $c$  appearing in this equation will be discussed below (Sect. 6).

We can see from equation (15) that the representation basis contains both integer and half-integer quasispins  $K$  and  $L$ . Equation (15) follows from the commutation relations

$$[K_0, \sigma_{\pm 1}] |\{K\}; d\rangle = -\frac{1}{2} \sigma_{\pm 1} |\{K\}; d\rangle \quad (16)$$

$$[L_0, \sigma_{\pm 1}] | \{K\}; d \rangle = -\frac{1}{2} \sigma_{\pm 1} | \{K\}; d \rangle \quad (17)$$

$$\begin{aligned} [K^2, \sigma_{\pm 1}] | \{K\}; d \rangle &= -\sigma_{\pm 1} \left( K_0 + \frac{1}{4} \right) | \{K\}; d \rangle \\ &= -\left( K + \frac{1}{4} \right) \sigma_{\pm 1} | \{K\}; d \rangle \end{aligned} \quad (18)$$

$$\begin{aligned} [L^2, \sigma_{\pm 1}] | \{K\}; d \rangle &= -\sigma_{\pm 1} \left( L_0 + \frac{1}{4} \right) | \{K\}; d \rangle \\ &= -\left( K + \frac{1}{4} \right) \sigma_{\pm 1} | \{K\}; d \rangle \end{aligned} \quad (19)$$

and

$$[-J_{56}, \sigma_{\pm 1}] | \{K\}; d \rangle = \pm \sigma_{\pm 1} | \{K\}; d \rangle. \quad (20)$$

### 5. Representation basis.

Now, we are in a position to construct explicitly all the MA states belonging to the representation basis and, consequently (employing in addition the  $K_-$  and  $L_-$  operators), all the representation basis. It is simply sufficient to act several times with the operators  $\sigma_{\pm 1}$ , on the state

$$|\max\rangle \equiv | \{K\}; d \rangle_{d=0}^{K=\Omega/2} \quad (21)$$

corresponding to the highest weight  $|J'_{12}, J'_{34}, J'_{56}\rangle = |\Omega, 0, 0\rangle$ . We obtain

$$\sigma_{-1}^m \sigma_{+1}^n |\max\rangle = C | \{K\}; d \rangle \quad (22)$$

with

$$K = \frac{\Omega}{2} - \frac{m}{2} - \frac{n}{2}, \quad (23)$$

$$d = n - m \quad (24)$$

and where  $C$  is a constant which will be calculated later.

It follows from equations (16-20) that the quantum numbers appearing on the right-hand side of equation (22) do not depend on the order of the operators  $\sigma_{+1}$  and  $\sigma_{-1}$  in this equation. Moreover, one can show that the constant  $C$  appearing in equation (22) does not depend on the order of  $\sigma_{+1}$  and  $\sigma_{-1}$ , either. It can easily be shown that  $[\sigma_{+1}, \sigma_{-1}] | \{K\}; d \rangle = 0$ . Explicitly one gets

$$\begin{aligned} \sigma_{+1} \sigma_{-1} | \{K\}; d \rangle &= [(2K)(2K+1)]^2 \times \\ &\times \langle \{K-1\}; d | T_{-+} T_{-+} | \{K\}; d \rangle | \{K-1\}; d \rangle \end{aligned}$$

and

$$\begin{aligned} \sigma_{-1} \sigma_{+1} | \{K\}; d \rangle &= [(2K)(2K+1)]^2 \times \\ &\times \langle \{K-1\}; d | T_{+-} T_{+-} | \{K\}; d \rangle | \{K-1\}; d \rangle \end{aligned}$$

which lead to the same result since  $T_{-+} T_{+-}$  commute.

Equations (23) and (24) imply that for a given value of  $K$  we obtain

$$-(\Omega - 2K) \leq d \leq \Omega - 2K. \quad (25)$$

On the other hand, equation (20) together with equations (23) and (24) imply that  $d$  is an integer changing in step of 2 between the limits indicated in equation (25).

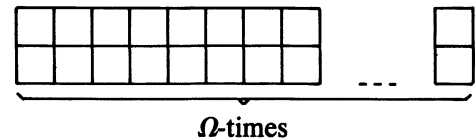
The total number of states obtained by acting with various powers of the step-down operators (18) on the highest weight state can now be calculated. The quantum number  $K$  varies between 0 and  $\frac{\Omega}{2}$ :

$$K = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{\Omega}{2}. \quad (26)$$

For each given value of  $K$  there exists  $\Omega - 2K + 1$  different values of  $d$  (cf. inequalities (25)), and, finally, there is a  $(2K+1)^2$  multiplicity for each pair  $(K, d)$  connected with the variations in the quantum numbers  $M_K$  and  $M_L$ . Thus the total number of states results as

$$\begin{aligned} \sum_{K=0, \frac{1}{2}, 1, \dots, \frac{\Omega}{2}} (\Omega - 2K + 1)(2K + 1)^2 &= \\ &= \frac{(\Omega + 1)(\Omega + 2)^2(\Omega + 3)}{12}. \end{aligned} \quad (27)$$

It is easy to see that this is exactly equal the dimension of the symmetric irreducible representation of the  $SO(6)$  group corresponding to the highest weight  $|\Omega, 0, 0\rangle$ . This can be shown either by the methods of Weyl [16] (see also Ref. [12], Chapter 14.2) or else by employing the equivalence of the  $SO(6)$  algebra to  $SU(4)$ . In fact (see [2]) the symmetric representation of  $SO(6)$  with highest weight  $|\Omega, 0, 0\rangle$  can be shown to correspond to the irreducible representation of  $SU(4)$  labelled by a Young diagram  $\{\Omega, \Omega\}$  i.e. which has obviously



the dimension given by (27) (see e.g. Close [17]). In this way, we have proved that the states (12) indeed span the total representation space.

### 6. Matrix elements of the group generators.

We shall now calculate explicitly the matrix elements of all the 15 generators of the group  $SO(6)$  in the basis of the symmetric irreducible representation discussed in a previous section. Matrix elements of the quasispin operators  $K$  and  $L$  are well known. For example :

$$L_+ |K, L; M_K, M_L; d\rangle_{L=K} = \\ = \sqrt{(K - M_L)(K + M_L + 1)} \\ \times |K, L; M_K, M_L + 1; d\rangle_{L=K}$$

etc. The three operators  $K_0$ ,  $L_0$  and  $J_{56}$  (or  $J_{12}$ ,  $J_{34}$  and  $J_{56}$ ) are diagonal with eigenvalues  $K'_0 = M_K$ ,  $L'_0 = M_L$  and  $J'_{56} = -d$ , respectively. We are finally left with the 8 tensor components (5a-h). In order to calculate their matrix elements we need first the norm of an arbitrary MA state (cf. Eq. (22)). Let us denote the norm of the state (22) by  $\mathcal{N}(m, n)$  i.e.

$$\mathcal{N}(m, n) = \langle \max | (\sigma_{+1}^\dagger)^n (\sigma_{-1}^\dagger)^m \sigma_{-1}^m \sigma_{+1}^n | \max \rangle. \quad (28)$$

As has been already mentioned in the previous section, one can show that the order of the operators  $\sigma_{+1}$  and  $\sigma_{-1}$  inside the matrix element (28) does not change the value of  $\mathcal{N}(m, n)$ . It can also be shown (see Appendix) that  $\mathcal{N}(m, n)$  satisfies two recursion relations which are considerably simplified and can be reduced to one recursion equation if one introduces the quantity

$$\rho(m, n) = (\Omega - m - n + 1)^{-3} \frac{\mathcal{N}(m+1, n)}{\mathcal{N}(m, n)}. \quad (29)$$

Then the recursion relations together with some limiting values become

$$\rho(m, 0) = (m+1)(\Omega - m)(\Omega - m + 1) \quad (30a)$$

$$\rho(0, n) = (\Omega - n)(\Omega + 1) \quad (30b)$$

$$\rho(m, n) = \{\Omega^2 - \Omega(4m + 2n - 1) + \\ + 3m^2 + n^2 + 2mn - 3m - 3n\} \\ + \rho(m-1, n) + \frac{\rho(n-1, m)}{(\Omega - m - n + 1)} \quad (30c)$$

$$(m \geq 1, n \geq 1).$$

It is not difficult to find the solution of the above set of equations

$$\rho(m, n) = (m+1)(\Omega - m - n)(\Omega - m + 1). \quad (31)$$

Having the above result, the rest of the calculation becomes standard. Owing to the SU(2) symmetry of the two quasispins we may use twice the Wigner-Eckart theorem

$$\langle K', K'; M'_K, M'_L; d' | T_{\kappa \lambda}^{1/2, 1/2, \delta} | K, K; M_K, M_L; d \rangle = \\ = \frac{\langle KM_K \frac{1}{2} \kappa | K' M'_K \rangle \langle KM_L \frac{1}{2} \lambda | K' M'_L \rangle}{(2K' + 1)} \\ \times \langle K' K' d' || T^{1/2, 1/2, \delta} || KKd \rangle \quad (32)$$

with  $d' = d + \delta$  and where the symbol  $\langle KM_K m | K' M' \rangle$  in equation (32) denotes a Clebsch-Gordan coefficient. The doubly reduced matrix element appearing in this equation can be calculated when the MA states are taken in equation (32) together with  $\kappa = \lambda = +\frac{1}{2}$  or  $\kappa = \lambda = -\frac{1}{2}$ . Then the matrix element can be either expressed directly by the norms  $\mathcal{N}(m, n)$  or equivalently by the quantity  $\rho(m, n)$ . In this case the left hand side of equation (32) can be calculated directly. Generally, tensor operators  $T_{\kappa \lambda}^{1/2, 1/2, \delta}$  can only change  $K$  quantum number by  $\pm \frac{1}{2}$  and quantum number  $d$  by  $\pm 1$ . In order to illustrate the procedure let us calculate a typical matrix element

$$\left\langle \left\{ K - \frac{1}{2} \right\}; d-1 | T_{-} | \{K\}; d \right\rangle.$$

It is useful to note that the tensor operator  $T_{-}$  when sandwiched between the two MA states can be replaced by a step-down operator  $\sigma_{-1}$  (up to a factor). Indeed, it follows from equation (14) that only the first term of  $\sigma_{-1}$  contributes in the matrix element given above. Now, the factor  $(2K_0 + 1)(2L_0 + 1)$  acting to the right gives a  $c$ -number equal to  $(2K + 1)^2$ . Thus we have

$$\left\langle \left\{ K - \frac{1}{2} \right\}; d-1 | T_{-} | \{K\}; d \right\rangle = \frac{1}{(2K + 1)^2} \times \\ \times \left\langle \left\{ K - \frac{1}{2} \right\}; d-1 | \sigma_{-1} | \{K\}; d \right\rangle. \quad (33)$$

Now, using equations (23), (24) and (28) we get

$$\left\langle \left\{ K - \frac{1}{2} \right\}; d-1 | T_{-} | \{K\}; d \right\rangle = \\ = \frac{1}{(2K + 1)^2} \frac{1}{\sqrt{\mathcal{N}(m+1, n)}} \left\langle \left\{ \frac{\Omega}{2} \right\}, 0 | (\sigma_{+1}^\dagger)^n (\sigma_{-1}^\dagger)^{m+1} \sigma_{-1} \sigma_{-1}^m \sigma_{+1}^n | \left\{ \frac{\Omega}{2} \right\}, 0 \right\rangle \frac{1}{\sqrt{\mathcal{N}(m, n)}} \\ = \frac{1}{(2K + 1)^2} \sqrt{\frac{\mathcal{N}(m+1, n)}{\mathcal{N}(m, n)}} = \frac{\sqrt{(\Omega - m - n + 1)^3 \rho(m, n)}}{(2K + 1)^2}. \quad (34)$$

On the other hand, we can use equation (32) directly for  $\kappa = \lambda = -\frac{1}{2}$ ,  $\delta = -1$ ,  $M_K = M_L = K$  and  $M'_K = M'_L = K' = K - \frac{1}{2}$ . Thus, we can determine the reduced matrix element

$$\langle K' K' d' \| T^{1/2 1/2 - 1} \| K K d \rangle$$

for  $K' = K - \frac{1}{2}$  and  $d' = d - 1$ . We can repeat the calculation for all possible non-vanishing matrix elements  $\left(K' = K \pm \frac{1}{2}, d' = d \pm 1\right)$  and substitute them into equation (32). The results for the non-vanishing matrix elements of all the tensors  $T_{\kappa \lambda}^{1/2 1/2 + 1}$  are given below

$$\begin{aligned} \left\langle K + \frac{1}{2}, K + \frac{1}{2}; M_K + \kappa, M_L + \lambda, d + 1 \mid T_{\kappa \lambda}^{1/2 1/2 + 1} \mid K, K; M_K, M_L, d \right\rangle = \\ = \frac{1}{2} \sqrt{\frac{(K + M_K \operatorname{sgn} \kappa + 1)(K + M_L \operatorname{sgn} \lambda + 1)(\Omega - 2K - d)(\Omega + 2K + d + 4)}{(2K + 1)(2K + 2)}} \end{aligned} \quad (35)$$

and

$$\begin{aligned} \left\langle K - \frac{1}{2}, K - \frac{1}{2}; M_K + \kappa, M_L + \lambda, d + 1 \mid T_{\kappa \lambda}^{1/2 1/2 + 1} \mid K, K; M_K, M_L, d \right\rangle = \\ = \frac{(-1)^{\kappa + \lambda}}{2} \sqrt{\frac{(K - M_K \operatorname{sgn} \kappa)(K - M_L \operatorname{sgn} \lambda)(\Omega - 2K + d + 2)(\Omega + 2K - d + 2)}{2K(2K + 1)}} \end{aligned} \quad (36)$$

where  $\operatorname{sgn} \kappa$  and  $\operatorname{sgn} \lambda$  denote the signs of  $\kappa$  and  $\lambda$ , respectively. Matrix elements of the tensor  $T_{\kappa \lambda}^{1/2 1/2 - 1}$  can be obtained from the above formulae by Hermitian conjugation (cf. Eq. (6)).

## 7. Relation to the Gel'fand Tsetlin basis.

Another possible decomposition of the  $\mathrm{SO}(6)$  irreducible representations is offered by the well-known Gel'fand Tsetlin method [3]. It is related to the chain

$$\mathrm{SO}(6) \supset \mathrm{SO}(5) \supset \mathrm{SO}(4) \supset \mathrm{SO}(3) \supset \mathrm{SO}(2). \quad (37)$$

The symmetric representations of  $\mathrm{SO}(6)$  can be labelled [4] by a single column of 5 numbers the first of which ( $\Omega$ ) characterizes the representation while the remaining 4 numbers ( $\Lambda, S_m, S$  and  $S_0$ ) determine the states of the basis. It seems convenient to define the  $\mathrm{SO}(3)$  subgroup of the chain (37) as a total quasispin

$$\mathbf{S} = \mathbf{K} + \mathbf{L} \quad (38)$$

where  $\mathbf{K}$  and  $\mathbf{L}$  have been already defined in the preceding sections. Then the irreducible symmetric representations of the  $\mathrm{SO}(4)$  may be defined by the maximum possible value of  $S$  (i.e.  $S_m = (K + L)_{L=K} = 2K$ ). Finally,  $\Lambda$  may characterize the irreducible symmetric representations of the subgroup  $\mathrm{SO}(5)$  (where  $S_m \leq \Lambda \leq \Omega$ ). The matrix elements of all the  $\mathrm{SO}(6)$  group generators can be calculated relatively easily with the method used in the case of symmetric representation (see Ref. [4]).

However, as already mentioned in the introduction, there exist some physical problems where it is more convenient to have  $J_{56}$  as a good quantum number. This is not the case in the Gel'fand Tsetlin

scheme [3]. Thus in this type of problem it is necessary to diagonalize the operator  $J_{56}$  explicitly in the latter basis. We thus arrive at the transformation

$$\begin{bmatrix} \Omega \\ d \\ S_m \\ S \\ S_0 \end{bmatrix} = \sum_{\Lambda=S_m}^{\Omega} C_d^{\Lambda}(\Omega, S_m) \begin{pmatrix} \Omega \\ \Lambda \\ S_m \\ S \\ S_0 \end{pmatrix} \quad (39)$$

where columns with parentheses denote the original Gel'fand Tsetlin representation basis while those with square brackets denote the new basis in which the operator  $(-J_{56})$  is diagonal with eigenvalue  $d$ . It follows from the Gel'fand Tsetlin rules that the operator  $J_{56}$  does not change the quantum numbers  $S_m, S$  and  $S_0$ . Thus the summation in equation (39) extends only over  $\Lambda$ . The eigenvalues occurring in the diagonalisation are  $d = 0, \pm 1, \pm 2, \dots$ . It has been also our numerical experience that they fulfil relations equivalent to inequalities (25) and that  $d$  changes in steps of 2 for given  $K$  (i.e.  $S_m$  and  $d$  are both even numbers or else both odd numbers, simultaneously). The transformation coefficients  $C_d^{\Lambda}(\Omega, S_m)$  have also to be determined numerically. All these properties become clear when we notice that there exists a simple relation

$$\begin{bmatrix} \Omega \\ d \\ S_m \\ S \\ S_0 \end{bmatrix} = \sum_{M_K, M_L} \langle K M_K L M_L | S S_0 \rangle \times \langle K, L; M_K, M_L, d \rangle_{L=K} \quad (40)$$

(where  $S_m = 2K$ )

connecting the new representation (39) with that

used throughout this paper (cf. Eqs. (8) and (9)). Nevertheless, the latter is much more convenient for practical calculations.

### 8. An example of physical application.

The decomposition of the symmetric irreducible representation discussed in this paper has been found [5] in the course of the calculation devoted to the coupled beta and gamma vibrations in deformed nuclei in the framework of an exactly solvable microscopic model. The model involves two degenerate single-particle levels with the degeneracy  $2\Omega$  for each level with pairing and quadrupole forces. The model Hamiltonian used in [5] can be expressed by the generators of SO(6) in a straightforward way

$$H = \varepsilon J_{34} - G(J_{13} + iJ_{23})(J_{13} - iJ_{23}) - 2\chi_0 q^2 J_{34}^2 - \chi_2 q^2 \{ (J_{35} + iJ_{36})(J_{35} - iJ_{36}) + (J_{35} - iJ_{36})(J_{35} + iJ_{36}) \}. \quad (41)$$

The first term describes the single-particle splitting between the two levels, the second term is a pairing force while the third and fourth terms correspond to the  $(\lambda, \mu) = (2, 0)$  and  $(\lambda, \mu) = (2, \pm 2)$  components of the quadrupole force, respectively. It is easy to see that both  $J_{56}$  and  $J_{12}$  are good quantum numbers i.e. they commute with the Hamiltonian. Thus, their values can be fixed, *a priori*, and the dimensions of submatrices to be diagonalized are considerably reduced as compared to the number given by equation (27). In fact the dimensions of the submatrices become of the order of  $\Omega^2$  instead of the  $\Omega^4$  dependence following from equation (27). The explicit formulae for the dimensions are given in [5].

We have performed the diagonalisation of the Hamiltonian (41) in two ways. First, by employing the Gel'fand Tsetlin method with the preliminary diagonalization of  $J_{56}$  followed by the diagonalisation of  $H$ . The other method consisted in using the representation discussed in this paper (cf. Sect. 3 to 5) with the explicit expressions for the matrix elements given in equations (35) and (36). The numerical results obtained for the energy eigenvalues and matrix elements of physical interest (i.e. quadrupole moments, transition probabilities etc.) were of course identical for both methods. However, the

latter has proved to be definitely superior in the numerical computations.

### 9. Conclusions.

We have demonstrated a new decomposition chain for the symmetric irreducible representations of the group SO(6) which differs essentially from that of the Wigner supermultiplet scheme. It has turned out to be possible to construct explicitly the representation basis and to calculate the matrix elements of the group generators in a closed and, in fact, very simple form. The scheme can, for example, be applied to the exact solution of the microscopic treatment of the pairing plus quadrupole two-level model in the case of the coupled beta and gamma vibrations in deformed nuclei [5]. The generalization of the above procedure into the case of arbitrary (i.e. not necessarily symmetric) irreducible representations of SO(6) seems to be also possible.

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### Appendix.

Instead of giving a full derivation for the calculation of the norm (28) and the resulting recursion relation (30) we shall only outline the main idea of the proof. We shall follow essentially an analogous calculation given by Hecht [18] for the simpler chain  $SO(5) \supset SU(2) \times SU(2)$ . We start with equation (28) for the arguments  $(m+1, n)$

$$\begin{aligned} \mathcal{N}(m+1, n) = & \langle \max | (\sigma_{+1}^\dagger)^n (\sigma_{-1}^\dagger)^{m+1} \sigma_{-1}^{m+1} \sigma_{+1}^n | \max \rangle = \\ & \langle \max | (\sigma_{+1}^\dagger)^n (\sigma_{-1}^\dagger)^m \sigma_{-1}^\dagger \sigma_{-1} \sigma_{-1}^m \sigma_{+1}^n | \max \rangle. \end{aligned} \quad (A.1)$$

Now the operator  $\sigma_{-1}^\dagger$  acts to the right (and in fact to the left, as well) on an MA state and consequently can be reduced to only one term  $T_{++}^\dagger$  multiplied from the left by  $(2K_0 + 1)(2L_0 + 1)$  which reduces to a number (cf. an analogous trick used already in the example given in Sect. 6). Thus :

$$\begin{aligned} \mathcal{N}(m+1, n) = & \langle \max | (\sigma_{+1}^\dagger)^n (\sigma_{-1}^\dagger)^m T_{++}^\dagger \sigma_{-1}^m \sigma_{+1}^n | \max \rangle (\Omega - m - n + 1)^2 = \\ & = (\Omega - m - n + 1)^2 \langle \max | (\sigma_{+1}^\dagger)^n (\sigma_{-1}^\dagger)^m \sigma_{-1} T_{++}^\dagger \sigma_{-1}^m \sigma_{+1}^n | \max \rangle \\ & + (\Omega - m - n + 1)^2 \langle \max | (\sigma_{+1}^\dagger)^n (\sigma_{-1}^\dagger)^m [T_{++}^\dagger, \sigma_{-1}] \sigma_{-1}^m \sigma_{+1}^n | \max \rangle. \end{aligned} \quad (A.2)$$

The last part of this equation is, of course, an identity. Let us call the first and second terms of the last part in equation (A.2) (I) and (II), respectively.

Calculating the commutator appearing in (II) and neglecting what vanishes when sandwiched in between two MA states we can express it by the

Casimir operator  $\hat{C}$  of the  $SO(6)$  group and some additional terms that we discuss below. The Casimir operator  $\hat{C}$  is defined as

$$\hat{C} = \sum_{a < b} J_{ab} J_{ab}. \quad (A.3)$$

It has an eigenvalue  $\Omega(\Omega + 4)$ . The additional terms following from the calculation of the commutator in (II) are either simple functions of  $J_{12}$ ,  $J_{34}$  and  $J_{56}$  and are thus diagonal or else form an expression

$$(T_{--} T_{++}^+ - T_{-+}^+ T_{++}^-) \quad (A.4)$$

that is to be sandwiched between the MA states as indicated by equation (A.2). We can now express (I) as well as both contributions coming from (A.4) by using the same trick as in section 6 and by observing that

$$T_{++}^+ \sigma_{-1}^m \sigma_{+1}^n |\max\rangle \sim \sigma_{-1}^{m-1} \sigma_{+1}^n |\max\rangle \quad (A.5)$$

and

$$T_{++}^- \sigma_{-1}^m \sigma_{+1}^n |\max\rangle \sim \sigma_{-1}^m \sigma_{+1}^{n-1} |\max\rangle \quad (A.6)$$

with proportionality coefficients depending on  $\mathcal{N}(m, n)$  and  $\mathcal{N}(m-1, n)$ , or  $\mathcal{N}(m, n)$  and  $\mathcal{N}(m, n-1)$ , respectively. Taking all the terms together we finally arrive at a recursion relation

$$\begin{aligned} \mathcal{N}(m+1, n) &= (\Omega - m - n + 1)^3 \times \\ &\times [\Omega^2 - \Omega(4m + 2n - 1) + 3m^2 \\ &+ 2mn + n^2 - 3m - 3n] \mathcal{N}(m, n) \\ &+ \frac{(\Omega - m - n + 1)^3}{(\Omega - m - n + 2)^3} (1 - \delta_{m,0}) \frac{\mathcal{N}^2(m, n)}{\mathcal{N}(m-1, n)} \\ &+ \frac{(\Omega - m - n + 1)^2}{(\Omega - m - n + 2)^3} (1 - \delta_{n,0}) \frac{\mathcal{N}^2(m, n)}{\mathcal{N}(m, n-1)}. \end{aligned} \quad (A.7)$$

Using relation (A.7) and its analogous for  $\mathcal{N}(m, n+1)$  one can calculate the first values of the norms. Further, by induction, one may easily show that

$$\mathcal{N}(m, n) = \mathcal{N}(n, m). \quad (A.8)$$

Now, using the definition (29) of  $\rho(m, n)$  (observe that  $\rho(m, n) \neq \rho(n, m)$ ) one can easily derive the recursion relations (30).

### References

- [1] WIGNER, E., *Phys. Rev.* **51** (1937) 106.
- [2] HECHT, K. T. and PANG, S. C., *J. Math. Phys.* **10** (1969) 1571.
- [3] GEL'FAND, J. M. and TSETLIN, M. L., *Dokl. Akad. Nauk SSSR* **71** (1950) 1017.
- [4] KRUMLINDE, J. and SZYMANSKI, Z., *Ann. Phys.* **79** (1973) 201.
- [5] PIEPENBRING, R., SILVESTRE-BRAC, B. and SZYMANSKI, Z., *Z. Phys. A* **318** (1984) 49.
- [6] KING, R. C., *Physica* **114A** (1982) 345.
- [7] KING, R. C. and EL-SHARKAWAY, N. G. I., *J. Phys. A Math. Gen.* **16** (1983) 3153.
- [8] SLANSKY, R., *Phys. Rep.* **79** (1981) 1.
- [9] LORENTE, M. and GRUBER, B., *J. Math. Phys.* **13** (1972) 1639.
- [10] AMAR, V., DOZZIO, U. and OLEARI, C., *J. Math. Phys.* **25** (1984) 2140.
- [11] RACAH, G., *Group Theory and Spectroscopy*, Princeton Lecture Notes : CERN preprint 61 (1961).
- [12] WYBOURNE, B. G., *Classical Groups for Physicists* (Wiley Interscience, New York) 1974.
- [13] BLACK, G. R. E. and WYBOURNE, B. G., *J. Phys. A Math. Gen.* **16** (1983) 2405.
- [14] KING, R. C., *J. Phys. A Math. Gen.* **8** (1975) 429.
- [15] BLACK, G. R. E., KING, R. C. and WYBOURNE, B. G., *J. Phys. A Math. Gen.* **16** (1983) 1555.
- [16] WEYL, H., *The classical groups* (Princeton U.P., Princeton, 1946).
- [17] CLOSE, F. E., *An introduction to quarks and partons* (Academic, New York) 1979.
- [18] HECHT, K. T., *Nucl. Phys.* **63** (1965) 177.